XXIX. *On the Analytical Theory of the Conic.* *By* Arthur Cayley, *F.R.S.*

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The decomposition into its linear factors of a decomposable quadric function cannot be effected in a symmetrical manner otherwise than by formulæ containing supernumerary arbitrary quantities; thus, for a binary quadric (which of course is always decomposable) we have

$$(a, b, c \between x, y)^2 = \frac{1}{(a, b, c \between x', y')^2} \text{ Prod. } \{(a, b, c \between x, y \between x', y') \pm \sqrt{ac-b^2}(xy'-x'y)\};$$

or the expression for a linear factor is

$$\frac{1}{\sqrt{(a, b, c \between x', y')^2}} \{(a, b, c \between x, y \between x', y') \pm \sqrt{ac-b^2}(xy'-x'y)\},$$

which involves the arbitrary quantities (x', y'). And this appears to be the reason why, in the analytical theory of the conic, the questions which involve the decomposition of a decomposable ternary quadric have been little or scarcely at all considered: thus, for instance, the expressions for the coordinates of the points of intersection of a conic by a line (or say the line-equations of the two ineunts), and the equations for the tangents (separate each from the other) drawn from a given point not on the conic, do not appear to have been obtained. These questions depend on the decomposition of a decomposable ternary quadric, which decomposition itself depends on that for the simplest case, when the quadric is a perfect square. Or we may say that in the first instance they depend on the transformation of a given quadric function $U=(*\between x, y, z)^2$ into the form W^2+V, where W is a linear function, given save as to a constant factor (that is, $W=0$ is the equation of a given line), and V is a decomposable quadric function, which is ultimately decomposed into its linear factors, $=QR$, so that we have $U=W^2+QR$. The formula for this purpose, which is exhibited in the eight different forms I, II, III, IV, I(bis), II(bis), III(bis), IV(bis), is the analytical basis of the whole theory; and the greater part of the memoir relates to the establishment of these forms.

The solution of the geometrical questions above referred to is (as shown in the memoir) involved in and given immediately by these forms. It is also shown that the formulæ are greatly simplified in the case *e. g.* of tangents drawn to a conic from a point in a conic having double contact with the first-mentioned conic, and that in this case they lead to the linear Automorphic Transformation of the ternary quadric. The memoir concludes with some formulæ relating to the case of two conics, which however is treated of in only a cursory manner.

Article Nos. 1 to 17, relating to a single conic.

1. The point-equation of the conic is

$$(a, b, c, f, g, h \between x, y, z)^2=0,$$

which expresses that the point (x, y, z) is an ineunt of the conic.

The line-equation of the same conic is

$$\begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi & a, & h, & g \\ \eta & h, & b, & f \\ \zeta & g, & f, & c \end{vmatrix}=0,$$

or putting

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H})=(bc-f^2, ca-g^2, ab-h^2, gh-af, hf-bg, fg-ch),$$

the line-equation is

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \xi, \eta, \zeta)^2=0,$$

which expresses that the line (ξ, η, ζ) (that is, the line the point-equation whereof is $\xi x+\eta y+\zeta z=0$) is a tangent of the conic. We are thus in the analytical theory of the conic concerned with the quadrics $(a, b, c, f, g, h \between x, y, z)^2$ and $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \xi, \eta, \zeta)^2$, which are the characteristics or *nilfactums* of these equations respectively.

2. I put also

$$\mathrm{K}=\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

or, what is the same thing,

$$\mathrm{K}=abc-af^2-bg^2-ch^2+2fgh.$$

3. It may be convenient to notice that when $(a, .. \between x, y, z)^2$ breaks up into factors, the conic the equation whereof is $(a, .. \between x, y, z)^2=0$, becomes a pair of lines; and that when $(a, .. \between x, y, z)^2$ is a perfect square, the conic becomes a pair of coincident lines, or say a *twofold* line. But a pair of lines, distinct or coincident, cannot be represented by a line-equation. The analytical formulæ presently given show that in the former case $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2$ is the square of a linear function, which equated to zero gives the line-equation of the point of intersection of the two lines, or node of the conic; and the equation $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2=0$ accordingly represents such point considered as a pair of coincident points, or say a *twofold* point. But in the latter case, where the conic is a twofold line, $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2$ is identically equal to zero, and the line-equation $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2=0$ is a mere identity $0=0$, thus ceasing to have any signification at all. And the like remarks apply to the conic as represented by the line-equation $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2=0$, the conic here breaking up into a pair of distinct or coincident points, &c.

4. It is proper to remark also that

$$(a, ... \between x', y', z' \between x, y, z)=0$$

is the equation of the polar of the point (x', y', z') in regard to the conic, and that

$$(\mathrm{A}, \ldots)(\xi', \eta', \zeta')(\xi, \eta, \zeta)=0$$

is the line-equation of the pole of the line (ξ', η', ζ'); or, what is the same thing, the point-coordinates of the pole are

$$\mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta' : \mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta' : \mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta'.$$

5. The inverse matrix is

$$\begin{pmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{pmatrix}^{-1}, = \frac{1}{\mathrm{K}}\begin{pmatrix} \mathrm{A}, & \mathrm{H}, & \mathrm{G} \\ \mathrm{H}, & \mathrm{B}, & \mathrm{F} \\ \mathrm{G}, & \mathrm{F}, & \mathrm{C} \end{pmatrix};$$

but it is convenient to disregard the factor $\frac{1}{\mathrm{K}}$, and speak of (A, B, C, F, G, H) as the inverse or reciprocal coefficients. The equation just written down implies the relations $\mathrm{A}a+\mathrm{H}h+\mathrm{G}g=\mathrm{K}$, $\mathrm{A}h+\mathrm{H}b+\mathrm{G}f=0$, &c., which may be arranged in two different ways as a system of nine equations.

6. We have also

$$(\mathrm{BC}-\mathrm{F}^2,\ \mathrm{CA}-\mathrm{G}^2,\ \mathrm{AB}-\mathrm{H}^2,\ \mathrm{GH}-\mathrm{AF},\ \mathrm{HF}-\mathrm{BG},\ \mathrm{FG}-\mathrm{CH})=\mathrm{K}(a, b, c, f, g, h),$$

and

$$\mathrm{ABC}-\mathrm{AF}^2-\mathrm{BG}^2-\mathrm{CH}^2+2\mathrm{FGH}=\mathrm{K}^2,$$

which are well-known theorems.

7. I notice also the theorem

$$(a, \ldots)(x, y, z)^2 . (a, \ldots)(x', y', z')^2-[(a, \ldots)(x, y, z)(x', y', z')]^2$$
$$=(\mathrm{A}, \ldots)(yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2,$$

which is much used in the sequel: it may be mentioned, in passing, that this is included in the more general theorem

$$\begin{vmatrix} (a, \ldots)(x, y, z)(l, m, n), & (a, \ldots)(x', y', z')(l, m, n) \\ (a, \ldots)(x, y, z)(l', m', n'), & (a, \ldots)(x', y', z')(l, m, n) \end{vmatrix}$$
$$=(\mathrm{A}, \ldots)(yz'-y'z,\ zx'-z'x,\ xy'-x'y)(mn'-m'n,\ nl'-n'l,\ lm'-l'm),$$

which is at once deducible from

$$\begin{vmatrix} \mathrm{L}l+\mathrm{M}m+\mathrm{N}n, & \mathrm{L}'l+\mathrm{M}'m+\mathrm{N}'n \\ \mathrm{L}l'+\mathrm{M}m'+\mathrm{N}n', & \mathrm{L}'l'+\mathrm{M}'m'+\mathrm{N}'n' \end{vmatrix}$$
$$=(\mathrm{MN}'-\mathrm{M}'\mathrm{N})(mn'-m'n)+(\mathrm{NL}'-\mathrm{N}'\mathrm{L})(nl'-n'l)+(\mathrm{LM}'-\mathrm{L}'\mathrm{M})(lm'-l'm),$$

by writing therein

$$(\mathrm{L}, \mathrm{M}, \mathrm{N})=(ax+hy+gz,\ hx+by+fz,\ gx+fy+cz),$$
$$(\mathrm{L}', \mathrm{M}', \mathrm{N}')=(ax'+hy'+gz',\ hx'+by'+fz',\ gx'+fy'+cz').$$

8. Suppose now that

$$(a, b, c, f, g, h \between x, y, z)^2$$

breaks up into factors, or say that we have

$$(a, b, c, f, g, h \between x, y, z)^2 = 2(\alpha x+\beta y+\gamma z)(\alpha' x+\beta' y+\gamma' z),$$

the values of the coefficients $(a, ..)$ then are

$$(a, b, c, f, g, h) = (2\alpha\alpha', 2\beta\beta', 2\gamma\gamma', \beta\gamma'+\beta'\gamma, \gamma\alpha'+\gamma'\alpha, \alpha\beta'+\alpha'\beta),$$

and forming from these the inverse coefficients (A, ..) and the discriminant K, we find

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H}) = -(\beta\gamma'-\beta'\gamma, \gamma\alpha'-\gamma'\alpha, \alpha\beta'-\alpha'\beta)^2.$$
$$\mathrm{K}=0.$$

9. The last-mentioned equation, $\mathrm{K}=0$, is the condition in order that $(a, .. \between x, y, z)^2$ may break up into factors; and when it does so, we have

$$(\mathrm{A}, .. \between \xi, \eta, \zeta)^2 = -[(\beta\gamma'-\beta'\gamma, \gamma\alpha'-\gamma'\alpha, \alpha\beta'-\alpha'\beta \between \xi, \eta, \zeta)]^2,$$

that is, $(a, .. \between x, y, z)^2$ breaking up into factors, $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2$ is a perfect square; and equating it to zero, we have

$$[(\beta\gamma'-\beta'\gamma, \gamma\alpha'-\gamma'\alpha, \alpha\beta'-\alpha'\beta \between \xi, \eta, \zeta)]^2=0;$$

which, (ξ, η, ζ) being line-coordinates, gives (as a twofold point) the point of intersection of the lines (α, β, γ), $(\alpha', \beta', \gamma')$, that is, the lines $\alpha x+\beta y+\gamma z=0$, $\alpha' x+\beta' y+\gamma' z=0$.

10. If $(a, .. \between x, y, z)^2$ is a perfect square, then $\alpha' : \beta' : \gamma' = \alpha : \beta : \gamma$; whence not only, as before, $\mathrm{K}=0$, but the coefficients (A, B, C, F, G, H) all vanish (this implies the first-mentioned condition, $\mathrm{K}=0$); and the line-equation $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2=0$ becomes the mere identity $0=0$.

11. Conversely if $\mathrm{K}=0$, then $(a, .. \between x, y, z)^2$ breaks up into factors; and if (A, B, C, F, G, H) all vanish, then $(a, .. \between x, y, z)^2$ is a perfect square. The conclusions stated *ante*, No. 3, are thus sustained.

12. I assume, first, that $(a, .. \between x, y, z)^2$ is a perfect square (No. 13); and secondly, that it breaks up into factors (No. 14); and I proceed to inquire how in the one case the root, and in the other case the factors, can be determined in a symmetrical form.

13. Considering the before-mentioned identical equation

$$(a, .. \between x, y, z)^2 . (a, .. \between x', y', z')^2 - [(a, .. \between x, y, z \between x', y', z')]^2 = (\mathrm{A}, .. \between yz'-y'z, zx'-z'x, xy'-x'y)^2,$$

if $(a, .. \between x, y, z)^2$ is a perfect square, then by what precedes, the right-hand side of the equation vanishes, and we have

$$(a, .. \between x, y, z)^2 = \frac{[(a, .. \between x, y, z \between x', y', z')]^2}{(a, .. \between x', y', z')^2};$$

and the root of $(a, .. \between x, y, z)^2$ is thus seen to be

$$= \pm \frac{(a, .. \between x, y, z \between x', y', z')}{\sqrt{(a, .. \between x', y', z')^2}},$$

an expression which involves the quantities (x', y', z'), the values whereof may be assumed at pleasure without altering the value of the expression. For instance, assuming for (x', y', z') the values (1, 0, 0), (0, 1, 0), (0, 0, 1) successively, the different values of the expression are

$$\frac{ax+hy+gz}{\sqrt{a}},\quad \frac{hx+by+fz}{\sqrt{b}},\quad \frac{gx+fy+cz}{\sqrt{c}}.$$

But if, as assumed, $(a, ..\between x, y, z)^2$ be a perfect square $=(\alpha x+\beta y+\gamma z)^2$, then

$$(a, b, c, f, g, h)=(\alpha^2, \beta^2, \gamma^2, \beta\gamma, \gamma\alpha, \alpha\beta),$$

and each of the foregoing values becomes equal to the root $\alpha x+\beta y+\gamma z$. It is somewhat singular that it is not possible to obtain symmetrical formulæ without employing in this manner supernumerary arbitrary quantities such as (x', y', z').

14. Next, if $(a, \ldots\between x, y, z)^2$, instead of being a perfect square, only breaks up into factors, then in the foregoing identical equation the right-hand side is a perfect square, and by the formula just obtained its value is

$$\frac{[(A, ..\between X, Y, Z\between yz'-y'z, zx'-z'x, xy'-x'y)]^2}{(A, ..\between X, Y, Z)^2},$$

where (X, Y, Z) are supernumerary arbitrary quantities. The identical equation then gives

$$(a, ..\between x, y, z)^2=\frac{1}{(a, \ldots\between x', y', z')^2}\left\{[(a, ..\between x, y, z\between x', y', z')]^2+\frac{[(A, ..\between X, Y, Z\between yz'-y'z, zx'-z'x, xy'-x'y)]^2}{(A, ..\between X, X, Z)^2}\right\},$$

and consequently

$$(a, ..\between x, y, z)^2=\frac{1}{(a, ..\between x', y', z')^2}\text{ Product of}$$

$$\left\{(a, ..\between x, y, z\between x', y', z')\pm\frac{(A, ..\between X, Y, Z\between yz'-y'z, zx'-z'x, xy'-x'y)}{\sqrt{-(A, ..\between X, Y, Z)^2}}\right\},$$

a formula which exhibits the decomposition of $(a, ..\between x, y, z)^2$ assumed to be a function which breaks up into factors; the formula contains the two sets of supernumerary arbitrary quantities (x', y', z') and (X, Y, Z). It will be remembered that (A, ..) denotes the system of inverse or reciprocal coefficients $(bc-f^2, \ldots)$.

15. Consider the formula

$$(a, b, c, f, g, h\between \eta\zeta'-\eta'\zeta, \zeta\xi'-\zeta'\xi, \xi\eta'-\xi'\eta)^2=(\mathrm{a, b, c, f, g, h}\between \xi', \eta', \zeta')^2,$$

which gives

$$\begin{aligned}
\mathrm{a}&=\ c\eta^2+b\zeta^2-2f\eta\zeta,\\
\mathrm{b}&=\ a\zeta^2+c\xi^2-2g\zeta\xi,\\
\mathrm{c}&=\ b\xi^2+a\eta^2-2h\xi\eta,\\
\mathrm{f}&=-a\eta\zeta-f\xi^2+g\xi\eta+h\xi\zeta,\\
\mathrm{g}&=-b\zeta\xi+f\xi\eta-g\eta^2+h\eta\zeta,\\
\mathrm{h}&=-c\xi\eta+f\xi\zeta+g\eta\zeta-h\zeta^2;
\end{aligned}$$

and from these we deduce

$$\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right)\!\!\big)\!\!\big(\xi, \eta, \zeta) = (0, 0, 0),$$

viz. $a\xi + h\eta + g\zeta = 0$, &c.

Also

$$(bc-f^2,\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch) = (\xi, \eta, \zeta)^2.(A, B, C, F, G, H)\!\!\big)\!\!\big(\xi, \eta, \zeta)^2,$$

that is,

$$bc-f^2 = \xi^2(A, B, C, F, G, H)\!\!\big)\!\!\big(\xi, \eta, \zeta)^2, \text{ \&c.}$$

Whence also

$$(bc-f^2, \ldots)\!\!\big)\!\!\big(l, m, n)^2 = (l\xi+m\eta+n\zeta)^2\,(A, \ldots)\!\!\big)\!\!\big(\xi, \eta, \zeta)^2,$$

and

$$(bc-f^2, \ldots)\!\!\big)\!\!\big(l, m, n)\!\!\big)\!\!\big(l', m', n') = (l\xi+m\eta+n\zeta)(l'\xi+m'\eta+n'\zeta)(A, \ldots)\!\!\big)\!\!\big(\xi, \eta, \zeta)^2;$$

and moreover

$$abc-af^2-bg^2-ch^2+2fgh=0.$$

16. The last equation shows that $(a, \ldots)\!\!\big)\!\!\big(\eta\zeta'-\eta'\zeta,\ \zeta\xi'-\zeta'\xi,\ \xi\eta'-\xi'\eta)^2$, considered as a function of (ξ', η', ζ'), breaks up into factors. Or since the expression is not altered by interchanging (ξ', η', ζ') and (ξ, η, ζ), the same expression, considered as a function of (ξ, η, ζ), breaks up into factors. It is in fact easy to see that any quantic whatever, $(*)\!\!\big)\!\!\big(\eta\zeta'-\eta'\zeta,\ \zeta\xi'-\zeta'\xi,\ \xi\eta'-\xi'\eta)^m$, considered as a function of (ξ, η, ζ), breaks up into linear factors; for in virtue of the equation $\xi'(\eta\zeta'-\eta'\zeta)+\eta'(\zeta\xi'-\zeta'\xi)+\zeta'(\xi\eta'-\xi'\eta)=0$, any one of the quantities $\eta\zeta'-\eta'\zeta$, $\zeta\xi'-\zeta'\xi$, $\xi\eta'-\xi'\eta$ can be expressed as a linear function of the other two; so that the quantic can be expressed as a linear function of any two of the three quantities; and *quà* homogeneous function of two quantities, it of course breaks up into factors, linear functions of these two quantities.

We may in all the formulæ interchange (x', y', z') and (x, y, z), writing (a', b', c', f', g', h') in the place of (a, b, c, f, g, h).

17. Putting, in like manner,

$$(A, B, C, F, G, H)\!\!\big)\!\!\big(yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2$$
$$= (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})\!\!\big)\!\!\big(x', y', z')^2,$$

so that

$$\begin{aligned} \mathfrak{A} &= Cy^2+Bz^2-2Fyz, \\ \mathfrak{B} &= Az^2+Cx^2-2Gzx, \\ \mathfrak{C} &= Bx^2+Ay^2-2Hxy, \\ \mathfrak{F} &= -Ayz-Fx^2+Gxy+Hzx, \\ \mathfrak{G} &= -Bzx+Fxy-Gy^2+Hyz, \\ \mathfrak{H} &= -Cxy+Fzx+Gyz-Hz^2, \end{aligned}$$

we obtain

$$\left(\begin{array}{ccc} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C} \end{array}\right)\!\!\big)\!\!\big(x, y, z) = (0, 0, 0),$$

viz. $\mathfrak{A}x+\mathfrak{H}y+\mathfrak{G}z=0$, &c.

Also

$$(\mathfrak{BC}-\mathfrak{F}^2,\ \mathfrak{CA}-\mathfrak{G}^2,\ \mathfrak{AB}-\mathfrak{H}^2,\ \mathfrak{GH}-\mathfrak{AF},\ \mathfrak{HF}-\mathfrak{BG},\ \mathfrak{FG}-\mathfrak{CH})$$
$$=(x,\ y,\ z)^2.\,\mathrm{K}(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2;$$

that is,

$$\mathfrak{BC}-\mathfrak{F}^2=x^2\mathrm{K}(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2,\ \&c.;$$

whence also

$$(\mathfrak{BC}-\mathfrak{F}^2,\ ..\between\lambda,\ \mu,\ \nu)^2=(\lambda x+\mu y+\nu z)^2.\,\mathrm{K}(a,\ ..\between x,\ y,\ z)^2,$$
$$(\mathfrak{BC}-\mathfrak{F}^2,\ ..\between\lambda,\ \mu,\ \nu\between\lambda',\ \mu',\ \nu')=(\lambda x+\mu y+\nu z)(\lambda'x+\mu'y+\nu'z).\,\mathrm{K}(a,\ ..\between x,\ y,\ z)^2;$$

and moreover

$$\mathfrak{ABC}-\mathfrak{AF}^2-\mathfrak{BG}^2-\mathfrak{CH}^2+2\mathfrak{FGH}=0.$$

The last equation shows that $(\mathrm{A},\ ..\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2$, considered as a function of $(x',\ y',\ z')$, breaks up into factors, or (what is the same thing) this expression, considered as a function of $(x,\ y,\ z)$, breaks up into factors; we may in all the formulæ interchange $(x,\ y,\ z)$ and $(x',\ y',\ z')$, writing $(\mathfrak{A}',\ \mathfrak{B}',\ \mathfrak{C}',\ \mathfrak{F}',\ \mathfrak{G}',\ \mathfrak{H}')$ in the place of $(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H})$.

Article Nos. 18 to 28, relating to a single conic in connexion with a point or line.

18. I apply the decomposition formula to the function $(\mathrm{A},\ ..\between yz'-y'z,\ ..)^2$, which, considered as a function of $(x,\ y,\ z)$, breaks up into factors. We have

$$(\mathrm{A},\ ..\between yz'-y'z,\ ..)^2=(\mathfrak{A}',\ ..\between x,\ y,\ z)^2$$
$$=\frac{1}{(\mathfrak{A}',\ ..\between l,\ m,\ n)^2}\ \text{Product of}$$
$$\left\{(\mathfrak{A}',\ ..\between l,\ m,\ n\between x,\ y,\ z)\pm\frac{(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between mz-ny,\ ..\between\lambda,\ \mu,\ \nu)}{\sqrt{-(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between\lambda,\ \mu,\ \nu)^2}}\right\}.$$

But we have

$$(\mathfrak{A}',\ ..\between l,\ m,\ n)^2=(\mathrm{A},\ ..\between mz'-ny',\ ..)^2,$$
$$(\mathfrak{A}',\ ..\between l,\ m,\ n\between x,\ y,\ z)=(\mathrm{A},\ ..\between mz'-ny',\ ..\between yz'-y'z,\ ..),$$
$$(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between mz-ny,\ ..\between\lambda,\ \mu,\ \nu)$$
$$=[x'(mz-ny)+y'(nx-lz)+z'(ly-mx)](\lambda x'+\mu y'+\nu z')\,\mathrm{K}(a,\ ..\between x',\ y',\ z')^2,$$
$$(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between\lambda,\ \mu,\ \nu)^2=\qquad(\lambda x'+\mu y'+\nu z')^2\mathrm{K}(a,\ ..\between x',\ y',\ z')^2,$$

and thence

$$\frac{(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between mz-ny,\ ..\between\lambda,\ \mu,\ \nu)}{\sqrt{-(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between\lambda,\ \mu,\ \nu)^2}}=\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix}\sqrt{-\mathrm{K}(a,\ ..\between x',\ y',\ z')^2},$$

whence we have

$$(\mathrm{A}, \ldots \between yz'-y'z, \ldots)^2 = \frac{1}{(\mathrm{A}, \ldots \between mz'-ny')^2} \text{ Product of}$$

$$(\mathrm{A}, \ldots \between mz'-ny', \ldots \between yz'-y'z, \ldots) \pm \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix} \sqrt{-\mathrm{K}(a, \ldots \between x', y', z')^2};$$

And the identical equation

$$(a, \ldots \between x, y, z)^2 . (a, \ldots \between x', y', z')^2 - [(a, \ldots \between x, y, z \between x', y', z')]^2 = (\mathrm{A}, \ldots \between yz'-y'z, \ldots)^2$$

now gives

$$(a, \ldots \between x, y, z)^2 = \text{ Quotient by } (a, \ldots \between x', y', z')^2 \text{ of}$$

$$\text{I.} \quad \left\{ \begin{array}{l} [(a, \ldots \between x, y, z \between x', y', z')]^2 \\ + \text{ Quotient by } (\mathrm{A}, \ldots \between mz'-ny', \ldots)^2 \text{ of Product} \\ (\mathrm{A}, \ldots \between mz'-ny', \ldots \between yz'-y'z, \ldots) \pm \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix} \sqrt{-\mathrm{K}(a, \ldots \between x', y', z')^2}, \end{array} \right.$$

where the Product part may also be written

$$\begin{array}{l} (a, \ldots \between l, m, n \between x', y', z') . (a, \ldots \between x, y, z \between x', y', z') \\ -(a, \ldots \between x', y', z')^2 \qquad . (a, \ldots \between x, y, z \between l, m, n) \\ \pm \sqrt{-\mathrm{K}(a, \ldots \between x', y', z')^2} \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix}. \end{array}$$

19. Writing in the formula I.

$$\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right\between x', y', z') = (\xi', \eta', \zeta'),$$

we have

$$(x', y', z') = \frac{1}{\mathrm{K}} \left(\begin{array}{ccc} \mathrm{A}, & \mathrm{H}, & \mathrm{G} \\ \mathrm{H}, & \mathrm{B}, & \mathrm{F} \\ \mathrm{G}, & \mathrm{F}, & \mathrm{C} \end{array}\right\between \xi', \eta', \zeta'),$$

and thence

$$\mathrm{K}(a, \ldots \between x', y', z') \qquad = (\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2$$

$$(a, \ldots \between x, y, z \between x', y', z') = \xi' x + \eta' y + \zeta' z.$$

Assume

$$(l, m, n) = (\nu\eta' - \mu\zeta', \ \lambda\zeta' - \nu\xi', \ \mu\xi' - \lambda\eta'),$$

then from the foregoing values of (x', y', z')

$$mz' - ny' = \frac{1}{\mathrm{K}} \left\{ (\lambda\zeta' - \nu\xi')(\mathrm{G}\xi' + \mathrm{F}\eta' + \mathrm{C}\zeta') - (\mu\xi' - \lambda\eta')(\mathrm{H}\xi' + \mathrm{B}\eta' + \mathrm{F}\zeta') \right\}$$

$$=\frac{1}{K}\Big\{\lambda[\xi'(A\xi'+H\eta'+G\zeta')+\eta'(H\xi'+B\eta'+F\zeta')+\zeta'(G\xi'+F\eta'+C\zeta')]$$
$$-\lambda\xi'(A\xi'+H\eta'+G\zeta')$$
$$-\mu\xi'(H\xi'+B\eta'+F\zeta')$$
$$-\nu\xi'(G\xi'+F\eta'+C\zeta'),$$

that is

$$mz'-ny'=\frac{1}{K}\Big\{\lambda(A,\ldots)(\xi',\eta',\zeta')^2-\xi'(A,\ldots)(\xi',\eta',\zeta')(\lambda,\mu,\nu)\Big\},$$

and similarly

$$nx'-lz'=\frac{1}{K}\Big\{\mu(A,\ldots)(\xi',\eta',\zeta')^2-\eta'(A,\ldots)(\xi',\eta',\zeta')(\lambda,\mu,\nu)\Big\},$$

$$ly'-mx'=\frac{1}{K}\Big\{\nu(A,\ldots)(\xi',\eta',\zeta')^2-\zeta'(A,\ldots)(\xi',\eta',\zeta')(\lambda,\mu,\nu)\Big\};$$

and thence

$$(A,H,G)(mz'-ny',\ldots)=$$
$$\frac{1}{K}\Big\{(A,H,G)(\lambda,\mu,\nu).(A,\ldots)(\xi',\eta',\zeta')^2$$
$$-(A,H,G)(\xi',\eta',\zeta').(A,\ldots)(\xi',\eta',\zeta')(\lambda,\mu,\nu)\Big\}$$

with the like equations, writing H, B, F and G, F, C in the place of A, H, G successively: and we then have

$$(A,\ldots)(mz'-ny',\ldots)^2$$
$$=\frac{1}{K}\Big\{(A,\ldots)(\lambda,\mu,\nu)(mz'-ny',\ldots).(A,\ldots)(\xi',\eta',\zeta')^2$$
$$-(A,\ldots)(\xi',\eta',\zeta')(mz'-ny',\ldots).(A,\ldots)(\xi',\eta',\zeta')(\lambda,\mu,\nu)\Big\}.$$

But the foregoing values of $mz'-ny'$, $nx'-lz'$, $ly'-mx'$ give also

$$(A,\ldots)(\lambda,\mu,\nu)(mz'-ny',\ldots)$$
$$=\frac{1}{K}\Big\{(A,\ldots)(\lambda,\mu,\nu)^2.(A,\ldots)(\xi',\eta',\zeta')^2-[(A,\ldots)(\lambda,\mu,\nu)(\xi',\eta',\zeta')]^2\Big\}.$$

$$(A,\ldots)(\xi',\eta',\zeta')(mz'-ny',\ldots)$$
$$=\frac{1}{K}\Big\{(A,\ldots)(\lambda,\mu,\nu)(\xi',\eta',\zeta').(A,\ldots)(\xi',\eta',\zeta')^2-(A,\ldots)(\xi',\eta',\zeta')^2.(A,\ldots)(\lambda,\mu,\nu)(\xi',\eta',\zeta')\Big\}=0.$$

So that

$$(A,\ldots)(mz'-ny',\ldots)^2$$
$$=\frac{1}{K^2}(A,\ldots)(\xi',\eta',\zeta')^2.\{(A,\ldots)(\lambda,\mu,\nu)^2.(A,\ldots)(\xi',\eta',\zeta')^2-[(A,\ldots)(\lambda,\mu,\nu)(\xi',\eta',\zeta')]^2\}$$
$$=\frac{1}{K}(A,\ldots)(\xi',\eta',\zeta')^2.(a,\ldots)(\nu\eta'-\mu\zeta',\ldots)^2.$$

Similarly,

$$(\mathrm{A}, ..)(mz'-ny', ..)(yz'-y'z, ..)$$
$$=\frac{1}{\mathrm{K}}\Big\{(\mathrm{A}, ..)(\lambda, \mu, \nu)(yz'-y'z, ..).(\mathrm{A}, ..)(\xi', \eta', \zeta')^2$$
$$-(\mathrm{A}, ..)(\xi', \eta', \zeta')(yz'-y'z, ..).(\mathrm{A}, ..)(\xi', \eta', \zeta')(\lambda, \mu, \nu)\Big\}.$$

But

$$\begin{aligned}(\mathrm{A}, ..)(\lambda, \mu, \nu)(yz'-y'z, ..)\\ = {} & (\mathrm{A}\lambda+\mathrm{H}\mu+\mathrm{G}\nu)(yz'-y'z)\\ & +(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)(zx'-z'x)\\ & +(\mathrm{G}\lambda+\mathrm{F}\mu+\mathrm{C}\nu)(xy'-x'y)\\ = {} & x[y'(\mathrm{G}\lambda+\mathrm{F}\mu+\mathrm{C}\nu)-z'(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)]\\ & +y[z'(\mathrm{A}\lambda+\mathrm{H}\mu+\mathrm{G}\nu)-x'(\mathrm{G}\lambda+\mathrm{F}\mu+\mathrm{C}\nu)]\\ & +z[x'(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)-y'(\mathrm{A}\lambda+\mathrm{H}\mu+\mathrm{G}\nu)],\end{aligned}$$

which, substituting for x', y', z' their values

$$(x', y', z')=\frac{1}{\mathrm{K}}\left(\begin{array}{ccc}\mathrm{A}, & \mathrm{H}, & \mathrm{G}\\ \mathrm{H}, & \mathrm{B}, & \mathrm{F}\\ \mathrm{G}, & \mathrm{F}, & \mathrm{C}\end{array}\right)(\xi', \eta', \zeta'),$$

becomes

$$=\frac{1}{\mathrm{K}}\Big\{\begin{aligned} & x[(\mathrm{BC}-\mathrm{F}^2)(\nu\eta'-\mu\zeta')+(\mathrm{FG}-\mathrm{CH})(\lambda\zeta'-\nu\xi')+(\mathrm{HF}-\mathrm{BG})(\mu\xi'-\lambda\eta')]\\ & +y[(\mathrm{FG}-\mathrm{CH})(\nu\eta'-\mu\zeta')+(\mathrm{CA}-\mathrm{G}^2)(\lambda\zeta'-\nu\xi')+(\mathrm{GH}-\mathrm{AF})(\mu\xi'-\lambda\eta')]\\ & +z\,(\mathrm{HF}-\mathrm{BG})(\nu\eta'-\mu\zeta')+(\mathrm{GH}-\mathrm{AF})(\lambda\zeta'-\nu\xi')+(\mathrm{AB}-\mathrm{H}^2)(\mu\xi'-\lambda\eta')\Big\},\end{aligned}$$

which is

$$=(a, ..)(x, y, z)(\nu\eta'-\mu\zeta', ..);$$

and by merely writing (ξ', η', ζ') in the place of (λ, μ, ν), we have

$$(\mathrm{A}, ..)(\xi', \eta', \zeta')(yz'-y'z, ..)=0;$$

so that we find

$$(\mathrm{A}, ..)(mz'-ny', ..)(yz'-y'z, ..)$$
$$=\frac{1}{\mathrm{K}}(\mathrm{A}, ..)(\xi', \eta', \zeta')^2.(a, ..)(x, y, z)(\nu\eta'-\mu\zeta', \lambda\zeta'-\nu\xi', \mu\xi'-\lambda\eta').$$

Now, writing the formula I. in the form

$$(a, ..)(x, y, z)^2=\text{Quotient by }\mathrm{K}(a, ..)(x', y', z')^2\text{ of}$$

$$\left\{\begin{array}{l}\mathrm{K}[(a, ..)(x, y, z)(x', y', z')]^2\\ +\text{Quotient by }\mathrm{K}(\mathrm{A}, ..)(mz'-ny', ..)^2\text{ of}\\ \mathrm{K}^2\{[(\mathrm{A}, ..)(mz'-ny', ..)(yz'-y'z, ..)]^2+\left|\begin{array}{ccc}x, & y, & z\\ x', & y', & z'\\ l, & m, & n\end{array}\right|^2\mathrm{K}(a, ..)(x, y, z)^2\},\end{array}\right.$$

the right-hand side is

$$=\text{Quotient by } (\mathrm{A},\ldots)(\xi',\eta',\zeta')^2 \text{ of}$$

$$\left\{\begin{array}{l} \mathrm{K}(\xi'x+\eta'y+\zeta'z)^2 \\ +\text{Quotient by } (a,\ldots)(\nu\eta'-\mu\zeta',\ldots)^2(\mathrm{A},\ldots)(\xi',\eta',\zeta')^2 \text{ of} \\ \{[(a,\ldots)(\nu\eta'-\mu\zeta',\ldots)(x,y,z).(\mathrm{A},\ldots)(\xi',\eta',\zeta')^2]^2+\Pi^2(\mathrm{A},\ldots)(\xi',\eta',\zeta')^2\}, \end{array}\right.$$

where

$$\Pi=\mathrm{K}\left|\begin{array}{ccc} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{array}\right|,$$

or, what is the same thing,

$$\Pi=\left|\begin{array}{ccc} x, & y, & z \\ \mathrm{K}x', & \mathrm{K}y', & \mathrm{K}z' \\ l, & m, & n \end{array}\right|=\left|\begin{array}{ccc} x & y & z \\ \mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta', & \mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta', & \mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta' \\ \nu\eta'-\mu\zeta', & \lambda\zeta'-\nu\xi', & \mu\xi'-\lambda\eta'. \end{array}\right|;$$

More simply, the right-hand side is

$$=\text{Quotient by } (\mathrm{A},\ldots)(\xi',\eta',\zeta')^2 \text{ of}$$

$$\left\{\begin{array}{l} \mathrm{K}(\xi'x+\eta'y+\zeta'z)^2 \\ +\text{Quotient by } (a,\ldots)(\nu\eta'-\mu\zeta',\ldots)^2 \text{ of} \\ \{[(a,\ldots)(\nu\eta'-\mu\zeta',\ldots)(x,y,z)]^2(\mathrm{A},\ldots)(\xi',\eta',\zeta')^2+\Pi^2\}; \end{array}\right.$$

Or restoring the left-hand side, and resolving into its linear factors the function in { }, we have

$$(a,\ldots)(x,y,z)^2=\text{Quotient by } (\mathrm{A},\ldots)(\xi',\eta',\zeta')^2 \text{ of}$$

II.

$$\left\{\begin{array}{l} \mathrm{K}(\xi'x+\eta'y+\zeta'z)^2 \\ +\text{Quotient by } (a,\ldots)(\nu\eta'-\mu\zeta',\ldots)^2 \text{ of Product} \\ \Pi\pm\sqrt{-(\mathrm{A},\ldots)(\xi',\eta',\zeta')^2}(a,\ldots)(\nu\eta'-\mu\zeta',\ldots)(x,y,z), \end{array}\right.$$

where Π has the value given above, which may also be written

$$\begin{aligned}\Pi= & \ (\mathrm{A},\ldots)(\xi',\eta',\zeta')(\lambda,\mu,\nu)(\xi'x+\eta'y+\zeta'z) \\ & -(\mathrm{A},\ldots)(\xi',\eta',\zeta')^2(\lambda x+\mu y+\nu z).\end{aligned}$$

20. We deduce at once the inverse or reciprocal formulæ

$$(\mathrm{A},\ldots)(\xi,\eta,\zeta)^2=\text{Quotient by } (\mathrm{A},\ldots)(\xi',\eta',\zeta')^2 \text{ of}$$

III.

$$\left\{\begin{array}{l} [(\mathrm{A},\ldots)(\xi,\eta,\zeta)(\xi',\eta',\zeta')]^2 \\ +\text{Quotient by } (a,\ldots)(\nu\eta'-\mu\zeta',\ldots)^2 \text{ of K into Product} \\ (a,\ldots)(\nu\eta'-\mu\zeta',\ldots)(\eta\zeta'-\eta'\zeta,\ldots)\pm\sqrt{-(\mathrm{A},\ldots)(\xi',\eta',\zeta')^2}\left|\begin{array}{ccc} \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \\ \lambda, & \mu, & \nu \end{array}\right|, \end{array}\right.$$

where the Product part may also be written

$$\text{Product}\left[\frac{1}{K}(A,..\between \xi', \eta', \zeta' \between \lambda, \mu, \nu).(A,..\between \xi', \eta', \zeta' \between \xi, \eta, \zeta) - \frac{1}{K}(A,..\between \xi', \eta', \zeta')^2 \quad .(A,..\between \lambda, \mu, \nu \between \xi, \eta, \zeta) \pm\sqrt{-(A,..\between \xi', \eta', \zeta')^2}\begin{vmatrix} \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \\ \lambda, & \mu, & \nu \end{vmatrix}\right].$$

21. And also

IV.

$$(A,..\between \xi, \eta, \zeta)^2 = \text{Quotient by } (a,..\between x', y', z')^2 \text{ of } \left\{\begin{array}{l} K(\xi x'+\eta y'+\zeta z')^2 \\ +\text{Quotient by } K(A,..\between ny'-mz',..)^2 \text{ of Product} \\ K\Phi\pm\sqrt{-K(a,..\between x', y', z')^2}(A,..\between ny'-mz',..\between \xi, \eta, \zeta), \end{array}\right.$$

where

$$\Phi = \begin{vmatrix} \xi & \eta & \zeta \\ ax'+hy'+gz', & hx'+by'+fz', & gx'+fy'+cz' \\ ny'-mz', & lz'-nx', & mx'-ly' \end{vmatrix},$$

which may also be written

$$\begin{aligned} = \;& (a,..\between x', y', z' \between l, m, n)(x'\xi+y'\eta+z'\zeta) \\ & -(a,..\between x', y', z')^2 \quad . \quad (l\xi+m\eta+n\zeta). \end{aligned}$$

22. The geometrical signification is obvious. The formulæ I. and II. each of them show that the equation

$$(a,..\between x, y, z)^2=0$$

of the conic may be written in the form

$$W^2+\frac{1}{M}QR=0,$$

where $Q=0$, $R=0$ are any two tangents of the conic, and $W=0$ is the line joining the points of contact, or chord of contact corresponding to the two tangents; viz., in the formula I. we have

$$W=(a,..\between x', y', z' \between x, y, z),$$

$$\left.\begin{array}{l} Q \\ R \end{array}\right\}=(A,..\between mz'-ny',..\between yz'-y'z,..)\pm\sqrt{-K(a,..\between x, y, z)^2}\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix}$$

(or for a different form of Q, R see the formula). The quantities (x', y', z') are the coordinates of the point of intersection of the two tangents, or pole of the chord of contact: (l, m, n) are supernumerary arbitrary quantities, the values whereof do not affect the result *. And in the formula II. we have

$$W = \xi' x + \eta' y + \zeta' z,$$

$$\left.\begin{matrix} Q \\ R \end{matrix}\right\} = \Pi \pm \sqrt{-(A, ..\between \xi', \eta', \zeta')^2} \, (a, ..\between \nu\eta' - \mu\zeta', ..\between x, y, z)$$

(for the value of Π see the formula). The quantities (ξ', η', ζ') are the line-coordinates of the chord of contact (viz. the point-equation of this line is $\xi' x + \eta' y + \zeta' z = 0$); (λ, μ, ν) are supernumerary arbitrary quantities.

23. In the like manner the formulæ III. and IV. each of them show that the line-equation

$$(A, ..\between \xi, \eta, \zeta)^2 = 0$$

of the conic may be written in the form

$$W^2 + \frac{1}{M} QR = 0,$$

where $Q=0$, $R=0$ are any two ineunts of the conic, and $W=0$ is the point of intersection of the corresponding tangents; viz. in the formula III. we have

$$W = (A, ...\between \xi', \eta', \zeta' \between \xi, \eta, \zeta),$$

$$\left.\begin{matrix} Q \\ R \end{matrix}\right\} = (a, ...\between \nu\eta' - \mu\zeta', ..) \pm \sqrt{-(A, ..\between \xi', \eta', \zeta')^2} \begin{vmatrix} \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \\ \lambda, & \mu, & \nu \end{vmatrix}$$

(for another form of Q, R see the formula).

The quantities ξ', η', ζ' are the line-coordinates of the line through the two ineunts, or chord of contact; (λ, μ, ν) are supernumerary arbitrary quantities; and so in the formula IV. we have

$$W = x'\xi + y'\eta + z'\zeta,$$

$$\left.\begin{matrix} Q \\ R \end{matrix}\right\} = K\Phi \pm \sqrt{-K(a, ..\between x', y', z')^2} \, (A, ..\between ny' - mz', ..\between \xi, \eta, \zeta)$$

(for the value of Φ see the formula), where x', y', z' are the point-coordinates of the intersection of tangents at the two ineunts, or pole of the chord of contact; (l, m, n) are supernumerary arbitrary quantities.

24. We may, instead of the supernumerary arbitrary quantities (l, m, n) of the formula I., introduce the quantities (λ, μ, ν), where

$$(l, m, n) = \frac{1}{K} \left(\begin{matrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{matrix} \right\between \lambda, \mu, \nu).$$

* In a different point of view, viz. if we consider the formula I. as a transformation of the function $(a, ...\between x, y, z)^2$, then (x', y', z') and (l, m, n) would be each of them supernumerary arbitrary quantities: and so in the other like cases.

This gives

$$\begin{aligned}
&(\mathrm{A},\ \mathrm{H},\ \mathrm{G}\between mz'-ny',\ ..)\\
&\quad =\mathrm{A}(mz'-ny')+\mathrm{H}(nx'-lz')+\mathrm{G}(ly'-mx')\\
&\quad =x'(\mathrm{H}n-\mathrm{G}m)+y'(\mathrm{G}l-\mathrm{A}n)+z'(\mathrm{A}m-\mathrm{H}l)\\
&\quad =\frac{1}{\mathrm{K}}\cdot\ x'[\mathrm{H}(\mathrm{G}\lambda+\mathrm{F}\mu+\mathrm{C}\nu)-\mathrm{G}(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)]\\
&\qquad\quad +y'[\mathrm{G}(\mathrm{A}\lambda+\mathrm{H}\mu+\mathrm{G}\nu)-\mathrm{A}(\mathrm{G}\lambda+\mathrm{F}\mu+\mathrm{C}\nu)]\\
&\qquad\quad +z'[\mathrm{A}(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)-\mathrm{H}(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)]\\
&\quad =\ x'(g\mu-h\nu)+y'(f\mu-b\nu)+z'(c\mu-f\nu)\\
&\quad =\ \mu(gx'+fy'+cz')-\nu(hx'+by'+fz').
\end{aligned}$$

We have thus the system

$$\begin{aligned}
(\mathrm{A},\ \mathrm{H},\ \mathrm{G}\between mz'-ny',\ ..)&=\mu(gx'+fy'+cz')-\nu(hx'+by'+fz'),\\
(\mathrm{H},\ \mathrm{B},\ \mathrm{F}\between mz'-ny',\ ..)&=\nu(ax'+hy'+gz')-\lambda(gx'+fy'+cz'),\\
(\mathrm{G},\ \mathrm{F},\ \mathrm{C}\between mz'-ny',\ ..)&=\lambda(hx'+by'+fz')-\mu(ax'+hy'+gz'),
\end{aligned}$$

and thence

$$(\mathrm{A},\ ..\between mz'-ny',\ ..\between yz'-y'z,\ ..)$$

$$=-\begin{vmatrix} yz'-y'z, & zx'-z'x, & xy'-x'y \\ ax'+hy'+gz', & hx'+by'+fz', & gx'+fy'+cz' \\ \lambda, & \mu, & \nu \end{vmatrix};$$

or observing that the term in λ is

$$-(zx'-z'x)(gx'+fy'+cz')+(xy'-x'y)(hx'+by'+fz'),$$

which is

$$\begin{aligned}
&=\ x(x'(ax'+hy'+gz')+y'(hx'+fy'+fz')+z'(gx'+fy'+cz'))\\
&\quad -x.x'(ax'+hy'+gz')\\
&\quad -y.x'(hx'+by'+fz')\\
&\quad -z.x'(gx'+fy'+cz')\\
&=-x'(a,\ ..\between x,\ y,\ z\between x',\ y',\ z')+x(a,\ ..\between x',\ y',\ z')^2,
\end{aligned}$$

with similar expressions for the terms in μ, ν, we have

$$\begin{aligned}
&(\mathrm{A},\ ..\between mz'-ny',\ ..\between yz'-y'z,\ ..)\\
&\quad =-(\lambda x'+\mu y'+\nu z').(a,\ ..\between x,\ y,\ z\between x',\ y',\ z')+(\lambda x+\mu y+\nu z).(a,\ ..\between x',\ y',\ z')^2\ ;
\end{aligned}$$

and so also

$$\begin{aligned}
&(\mathrm{A},\ ..\between mz'-ny',\ ..)^2\\
&\quad =-(\lambda x'+\mu y'+\nu z').(a,\ ..\between l,\ m,\ n\between x',\ y',\ z')+(\lambda l+\mu m+\nu n).(a,\ ..\between x',\ y',\ z')^2,
\end{aligned}$$

where

$$\begin{aligned}
(a,\ ..\between l,\ m,\ n\between x',\ y',\ z')&=\lambda x'+\mu y'+\nu z',\\
\lambda l+\mu m+\nu n\qquad&=\frac{1}{\mathrm{K}}(\mathrm{A},\ ..\between \lambda,\ \mu,\ \nu)^2,
\end{aligned}$$

so that

$$(\mathrm{A}, \ldots \between mz'-ny', \ldots)^2 = -(\lambda x'+\mu y'+\nu z')^2 + \frac{1}{\mathrm{K}}(\mathrm{A}, \ldots \between \lambda, \mu, \nu)^2.(a, \ldots \between x', y', z')^2.$$

Moreover,

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix} = \frac{1}{\mathrm{K}}\Big\{(\mathrm{A}\lambda+\mathrm{H}\mu+\mathrm{G}\nu)(yz'-y'z)+(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)(zx'-z'x)+(\mathrm{G}\lambda+\mathrm{F}\mu+\mathrm{C}\nu)(xy'-x'y)\Big\},$$

which is

$$=\frac{1}{\mathrm{K}}(\mathrm{A}, \ldots \between \lambda, \mu, \nu \between yz'-y'z, \ldots);$$

and hence instead of the formula I. we have

$$\text{I. (bis)}\quad (a, \ldots \between x, y, z)^2 = \text{Quotient by } (a, \ldots \between x', y', z')^2 \text{ of } \left\{\begin{array}{l} [(a, \ldots \between x, y, z \between x', y', z')]^2 \\ +\text{Quotient by } +(\mathrm{A}, \ldots \between \lambda, \mu, \nu)^2(a, \ldots \between x', y', z')^2 - \mathrm{K}(\lambda x'+\mu y'+\nu z')^2 \text{ of K Product} \\ \left\{\begin{array}{l} (\lambda x'+\mu y'+\nu z').(a, \ldots \between x, y, z \between x', y', z') - (\lambda x+\mu y+\nu z).(a, \ldots \between x', y', z')^2 \\ \pm\frac{1}{\mathrm{K}}\sqrt{-\mathrm{K}(a, \ldots \between x', y', z')^2}(\mathrm{A}, \ldots \between \lambda, \mu, \nu \between yz'-y'z, \ldots). \end{array}\right\} \end{array}\right.$$

25. If, in like manner, in the formula II. we introduce, instead of (λ, μ, ν), the new quantities (l, m, n), where

$$(\lambda, \mu, \nu) = \left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array} \between l, m, n\right),$$

or, what is the same thing,

$$(l, m, n) = \frac{1}{\mathrm{K}}\left(\begin{array}{ccc} \mathrm{A}, & \mathrm{H}, & \mathrm{G} \\ \mathrm{H}, & \mathrm{B}, & \mathrm{F} \\ \mathrm{G}, & \mathrm{F}, & \mathrm{C} \end{array} \between \lambda, \mu, \nu\right),$$

then we have

$$\begin{aligned} (a, h, g \between \nu\eta'-\mu\zeta', \ldots) &= n\,(\mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta') - m(\mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta'), \\ (h, b, f \between \nu\eta'-\mu\zeta', \ldots) &= l\,(\mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta') - n\,(\mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta'), \\ (g, f, c \between \nu\eta'-\mu\zeta', \ldots) &= m(\mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta') - l\,(\mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta'); \end{aligned}$$

and thence

$$\begin{aligned} (a, \ldots \between \nu\eta'-\mu\zeta', \ldots \between x, y, z) &= \begin{vmatrix} x & , & y & , & z \\ \mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta', & & \mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta', & & \mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta' \\ l & , & m & , & n \end{vmatrix} \\ &= (\mathrm{A}, \ldots \between mz-ny, \ldots \between \xi', \eta', \zeta'), \end{aligned}$$

$$\begin{aligned} (a, \ldots \between \nu\eta'-\mu\zeta', \ldots)^2 &= \frac{1}{\mathrm{K}}\Big\{(\mathrm{A}, \ldots \between \lambda, \mu, \nu)^2.(\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2 - [(\mathrm{A}, \ldots \between \lambda, \mu, \nu \between \xi', \eta', \zeta')]^2\Big\} \\ &= (a, \ldots \between l, m, n)^2.(\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2 - \mathrm{K}(l\xi'+m\eta'+n\zeta')^2; \end{aligned}$$

$$
V = \begin{vmatrix} x \ , & y \ , & z \\ A\xi'+H\eta'+G\zeta', & H\xi'+B\eta'+F\zeta', & G\xi'+F\eta'+C\zeta' \\ \nu\eta'-\mu\zeta' \ , & \lambda\zeta'-\nu\xi' \ , & \mu\xi'-\lambda\eta' \end{vmatrix}
$$

$$
\begin{aligned}
&= (\nu\eta'-\mu\zeta').\,y(G\xi'+F\eta'+C\zeta')-z(H\xi'+B\eta'+F\zeta') \\
&\quad (\lambda\zeta'-\nu\xi').\,z(A\xi'+H\eta'+G\zeta')-x(G\xi+F\eta'+C\zeta') \\
&\quad (\mu\xi'-\lambda\eta').\,x(H\xi'+B\eta'+F\zeta')-y(A\xi'+H\eta'+G\zeta')
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda\{(\xi'x+\eta'y+\zeta'z)(A\xi'+H\eta'+G\zeta')-x(A,\ldots)\!(\xi',\eta',\zeta')^2\} \\
&\quad +\mu\{(\xi'x+\eta'y+\zeta'z)(H\xi'+B\eta'+F\zeta')-y(A,\ldots)\!(\xi',\eta',\zeta')^2\} \\
&\quad +\nu\{(\xi'x+\eta'y+\zeta'z)(G\xi'+F\eta'+C\zeta')-z(A,\ldots)\!(\xi',\eta',\zeta')^2\}
\end{aligned}
$$

$$
=(\xi'x+\eta'y+\zeta'z).(A,\ldots)\!(\lambda,\mu,\nu)\!(\xi',\eta',\zeta')-(\lambda x+\mu y+\nu z).(A,\ldots)\!(\xi',\eta',\zeta')^2
$$

$$
=K(l\xi'+m\eta'+n\zeta').(\xi'x+\eta'y+\zeta'z)-(a,\ldots)\!(l,m,n)\!(x,y,z).(A,\ldots)\!(\xi',\eta',\zeta')^2;
$$

and the formula II. thus becomes

$$
(a,\ldots)\!(x,y,z)^2 = \text{Quotient by } (A,\ldots)\!(\xi',\eta',\zeta')^2 \text{ of}
$$

II. (bis)

$$
\left\{\begin{array}{l}
K(\xi'x+\eta'y+\zeta'z)^2 \\
+\text{Quotient by } (a,\ldots)\!(l,m,n)^2.(A,\ldots)\!(\xi',\eta',\zeta')^2-K(l\xi'+m\eta'+n\zeta')^2 \text{ of Product} \\
\left\{\begin{array}{l} K(l\xi'+m\eta'+n\zeta').(\xi'x+\eta'y+\zeta'z)-(a,\ldots)\!(l,m,n)\!(x,y,z).(A,\ldots)\!(\xi',\eta',\zeta')^2 \\ \pm\sqrt{-(A,\ldots)\!(\xi',\eta',\zeta')^2}(A,\ldots)\!(mz-ny,\ldots)\!(\xi',\eta',\zeta'). \end{array}\right\}
\end{array}\right.
$$

26. And from these we at once deduce the inverse or reciprocal formulæ

$$
(A,\ldots)\!(\xi,\eta,\zeta)^2 = \text{Quotient by } (A,\ldots)\!(\xi',\eta',\zeta')^2 \text{ of}
$$

III. (bis)

$$
\left\{\begin{array}{l}
[(A,\ldots)\!(\xi,\eta,\zeta)\!(\xi',\eta',\zeta')]^2 \\
+\text{Quotient by } (a,\ldots)\!(l,m,n)^2.(A,\ldots)\!(\xi',\eta',\zeta')^2-K(l\xi'+m\eta'+n\zeta')^2 \text{ of K into Product} \\
\left\{\begin{array}{l} (l\xi'+m\eta'+n\zeta')(A,\ldots)\!(\xi,\eta,\zeta)\!(\xi',\eta',\zeta')-(l\xi+m\eta+n\zeta).(A,\ldots)\!(\xi',\eta',\zeta')^2 \\ \pm\sqrt{-(A,\ldots)\!(\xi',\eta',\zeta')^2}(a,\ldots)\!(l,m,n)\!(\eta\zeta'-\eta'\zeta,\ldots), \end{array}\right\}
\end{array}\right.
$$

27. And

$$
(A,\ldots)\!(\xi,\eta,\zeta)^2 = \text{Quotient by } (a,\ldots)\!(x',y',z')^2 \text{ of}
$$

IV. (bis)

$$
\left\{\begin{array}{l}
K(x'\xi+y'\eta+z'\zeta)^2 \\
+\text{Quotient by } (A,\ldots)\!(\lambda,\mu,\nu)^2.(a,\ldots)\!(x',y',z')^2-K(\lambda x'+\mu y'+\nu z')^2 \text{ of Product} \\
\left\{\begin{array}{l} K(\lambda x'+\mu y'+\nu z').(x'\xi+y'\eta+z'\zeta)-(A,\ldots)\!(\xi,\eta,\zeta)\!(l,m,n).(a,\ldots)\!(x',y',z')^2 \\ \pm\sqrt{-K(a,\ldots)\!(x',y',z')^2}(a,\ldots)\!(x',y',z')\!(\mu\zeta-\nu\eta,\ldots), \end{array}\right\}
\end{array}\right.
$$

which four formulæ have the same geometrical significations with the original four formulæ to which they correspond respectively.

28. The eight formulæ become all of them the same or very similar for the quadric form $(a, \ldots \between x, y, z)^2 = x^2+y^2+z^2$, which of course implies $(A, \ldots \between \xi, \eta, \zeta)^2 = \xi^2+\eta^2+\zeta^2$. Thus selecting any one of them at pleasure, *e. g.* the formula II. (bis), this becomes

$$\begin{aligned}
&\{(x^2+y^2+z^2)(\xi'^2+\eta'^2+\zeta'^2)-(\xi'x+\eta'y+\zeta'z)^2\} \\
\times &\{(l^2+m^2+n^2)(\xi'^2+\eta'^2+\zeta'^2)-(l\xi'+m\eta'+n\zeta')^2\} \\
= &\{(l\xi'+m\eta'+n\zeta')(x\xi'+m\eta'+n\zeta')-(lx+my+nz)(\xi'^2+\eta'^2+\zeta'^2)\}^2 \\
&+(\xi'^2+\eta'^2+\zeta'^2)\begin{vmatrix} \xi', & \eta', & \zeta' \\ x, & y, & z \\ l, & m, & n \end{vmatrix}^2,
\end{aligned}$$

where the terms independent of $\xi'^2+\eta'^2+\zeta'^2$ destroy each other. Omitting these terms, and dividing by $\xi'^2+\eta'^2+\zeta'^2$, the resulting equation is found to be

$$\begin{vmatrix} \xi', & \eta', & \zeta' \\ x, & y, & z \\ l, & m, & n \end{vmatrix}^2 = \begin{vmatrix} \xi'^2+\eta'^2+\zeta'^2, & \xi'x+\eta'y+\zeta'z, & \xi'l+\eta'm+\zeta'n \\ x\xi'+y\eta'+z\zeta', & x^2+y^2+z^2, & xl+ym+zn \\ l\xi'+m\eta'+n\zeta', & lx+my+nz, & l^2+m^2+n^2, \end{vmatrix}$$

which is a well-known identical equation.

Article Nos. 29 to 33, relating to a single conic in connexion with an ineunt or a tangent of a conic of double contact.

29. The formulæ assume a very simple form when the point of intersection of the two tangents, or the line of junction of the two ineunts of the conic, is an ineunt or a tangent of a conic having double contact with the first-mentioned conic. Thus, if to the conic

$$(a, \ldots \between x, y, z)^2 = 0$$

tangents are drawn from a point (x', y', z') of the conic

$$(a, \ldots \between x, y, z)^2 + (\xi'x+\eta'y+\zeta'z)^2 = 0,$$

then we have

$$(a, \ldots \between x', y', z')^2 = -(\xi'x'+\eta'y'+\zeta'z')^2;$$

and using the form I. (bis), and putting therein (ξ', η', ζ') in the place of the arbitrary quantities (λ, μ, ν), the equation of the tangent divides out by $\xi'x'+\eta'y'+\zeta'z'$, and omitting this factor it becomes

$$\begin{aligned}
&(a, \ldots \between x', y', z' \between x, y, z) + (\xi'x'+\eta'y'+\zeta'z')(\xi'x+\eta'y+\zeta'z) \\
&\pm \frac{1}{\sqrt{K}}(A, \ldots \between \xi', \eta', \zeta' \between yz'-y'z, zx'-z'x, xy'-x'y) = 0,
\end{aligned}$$

which is of the form

$$\left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array} \between x', y', z' \between x, y, z \right) = 0,$$

where the matrix

$$\left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma \\ \alpha'', & \beta'', & \gamma'' \end{array}\right) \text{ is } =$$

$$\left|\begin{array}{ccc} a+\xi'^2 & , h+\xi'\eta'+\frac{1}{\sqrt{K}}(G\xi'+F\eta'+C\zeta'), & g+\xi'\zeta'-\frac{1}{\sqrt{K}}(H\xi'+B\eta'+F\zeta') \\ h+\xi'\eta'-\frac{1}{\sqrt{K}}(G\xi'+F\eta'+C\zeta'), & b+\eta'^2 & , f+\eta'\zeta'+\frac{1}{\sqrt{K}}(A\xi'+H\eta'+G\zeta') \\ g+\xi'\zeta'+\frac{1}{\sqrt{K}}(H\xi'+B\eta'+F\zeta'), & f+\eta'\zeta'-\frac{1}{\sqrt{K}}(A\xi'+H\eta'+G\zeta'), & c+\zeta'^2. \end{array}\right.$$

30. But instead of further developing these formulæ, I prefer to consider the formulæ which give the points of contact of the tangents in question, viz. the ineunts of the conic $(a, .. \between x, y, z)^2=0$, or the tangents through the point (x', y', z') of the conic $(a, .. \between x, y, z)^2+(\xi'x+\eta'y+\zeta'z)^2=0$.

We have as before

$$(a, .. \between x', y', z')^2=-(\xi'x'+\eta'y'+\zeta'z')^2,$$

and using the formula IV.(bis) and writing therein (ξ', η', ζ') in the place of the arbitrary quantities (λ, μ, ν), the equation contains the factor $\xi'x'+\eta'y'+\zeta'z'$, and dividing by this factor, and by K, the line-equation of the ineunt is

$$x'\xi+y'\eta+z'\zeta+\frac{1}{K}(x'\xi'+y'\eta'+z'\zeta').(A, .. \between \xi', \eta', \zeta' \between \xi, \eta, \zeta)$$

$$\pm\frac{1}{\sqrt{K}}(a, .. \between x', y', z' \between \eta\zeta'-\eta'\zeta, ..)=0.$$

Selecting the positive sign, the coordinates of the corresponding ineunt are

$$x'+\frac{1}{K}(x'\xi'+y'\eta'+z'\zeta')(A\xi'+H\eta'+G\zeta')+\frac{1}{\sqrt{K}}\left\{\eta'(gx'+fy'+cz')-\zeta'(hx'+by'+fz')\right\},$$

$$y'+\frac{1}{K}(x'\xi'+y'\eta'+z'\zeta')(H\xi'+B\eta'+F\zeta')+\frac{1}{\sqrt{K}}\left\{\zeta'(ax'+hy'+gz')-\xi'(gx'+fy'+cz')\right\},$$

$$z'+\frac{1}{K}(x'\xi'+y'\eta'+z'\zeta')(G\xi'+F\eta'+C\zeta')+\frac{1}{\sqrt{K}}\left\{\xi'(hx'+by'+fz')-\eta'(ax'+hy'+gz')\right\};$$

and taking (X, Y, Z) for the coordinates of the ineunt in question, and putting for shortness

$$\alpha=1-\frac{1}{\sqrt{K}}(g\eta'-h\zeta'),\quad \beta=-\frac{1}{\sqrt{K}}(f\eta'-b\zeta'),\quad \gamma=-\frac{1}{\sqrt{K}}(c\eta'-f\zeta'),$$

$$\alpha'=-\frac{1}{\sqrt{K}}(a\zeta'-g\xi'),\quad \beta'=1-\frac{1}{\sqrt{K}}(h\zeta'-f\xi'),\quad \gamma'=-\frac{1}{\sqrt{K}}(g\zeta'-c\xi'),$$

$$\alpha''=-\frac{1}{\sqrt{K}}(h\xi'-a\eta'),\quad \beta''=-\frac{1}{\sqrt{K}}(b\xi'-h\eta'),\quad \gamma''=1-\frac{1}{\sqrt{K}}(f\xi'-g\eta'),$$

we may write

$$(1+\mathrm{P})\mathrm{X}=(2-\alpha)x' \quad -\beta y' \quad -\gamma z' \quad +\frac{1}{\mathrm{K}}(\mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$(1+\mathrm{P})\mathrm{Y}= \quad -\alpha'x'+(2-\beta')y'-\gamma'z' \quad +\frac{1}{\mathrm{K}}(\mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$(1+\mathrm{P})\mathrm{Z}= \quad \alpha''x'' \quad -\beta''y'+(2-\gamma'')z'+\frac{1}{\mathrm{K}}(\mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

where P, which is arbitrary, may be put

$$=\frac{1}{\mathrm{K}}(\mathrm{A},\,..\,\between\xi',\,\eta',\,\zeta')^2.$$

31. These equations then give

$$(x',\,y',\,z')=\left(\begin{array}{ccc}\alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma''\end{array}\right\between\mathrm{X},\,\mathrm{Y},\,\mathrm{Z}),$$

which can be verified without difficulty by reversing the process; and we have thus the coordinates (X, Y, Z) in terms of $(x',\,y',\,z')$, and reciprocally.

32. If $(\mathrm{X}_{,},\,\mathrm{Y}_{,},\,\mathrm{Z}_{,})$ are the coordinates of the other ineunt, we have, it is clear,

$$(x',\,y',\,z')=\left(\begin{array}{ccc}2-\alpha, & -\beta, & -\gamma \\ -\alpha', & 2-\beta', & -\gamma' \\ -\alpha'', & -\beta'', & 2-\gamma''\end{array}\right\between\mathrm{X}_{,},\,\mathrm{Y}_{,},\,\mathrm{Z}_{,});$$

or substituting for $(x',\,y',\,z')$ their values in terms of (X, Y, Z),

$$(2\mathrm{X}_{,},\,2\mathrm{Y}_{,},\,2\mathrm{Z}_{,})=\left(\begin{array}{ccc}\alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma''\end{array}\right\between\mathrm{X}+\mathrm{X}_{,},\,\mathrm{Y}+\mathrm{Y}_{,},\,\mathrm{Z}+\mathrm{Z}_{,}),$$

so that $(\mathrm{X}+\mathrm{X}_{,},\,\mathrm{Y}+\mathrm{Y}_{,},\,\mathrm{Z}+\mathrm{Z}_{,})$ are the same linear functions of $2\mathrm{X}_{,}$, $2\mathrm{Y}_{,}$, $2\mathrm{Z}_{,}$ that (X, Y, Z) are of $(x',\,y',\,z')$; that is, we have

$$\tfrac{1}{2}(1+\mathrm{P})(\mathrm{X}+\mathrm{X}_{,})=(2-\alpha)\mathrm{X}_{,}- \quad \beta\,\mathrm{Y}_{,}- \quad \gamma\,\mathrm{Z}_{,}+\frac{1}{\mathrm{K}}(\mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$\tfrac{1}{2}(1+\mathrm{P})(\mathrm{X}+\mathrm{X}_{,})= \quad -\alpha'\,\mathrm{X}_{,}+(2-\beta')\mathrm{Y}_{,}- \quad \gamma'\,\mathrm{Z}_{,}+\frac{1}{\mathrm{K}}(\mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$\tfrac{1}{2}(1+\mathrm{P})(\mathrm{X}+\mathrm{X}_{,})= \quad -\alpha''\,\mathrm{X}_{,}- \quad \beta''\,\mathrm{Y}_{,}+(2-\gamma'')\mathrm{Z}_{,}+\frac{1}{\mathrm{K}}(\mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

which equations may be written

$$(1+\mathrm{P})(\mathrm{X},\,\mathrm{Y},\,\mathrm{Z})=\left(\begin{array}{ccc}\mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}' \\ \mathrm{a}'', & \mathrm{b}'', & \mathrm{c}''\end{array}\right\between\mathrm{X}_{,},\,\mathrm{Y}_{,},\,\mathrm{Z}_{,}),$$

where the values of the coefficients are

$$
\begin{array}{lll}
1-\frac{2}{\sqrt{K}}(g\eta'-h\zeta')+\frac{1}{K}(A\xi'^2-B\eta'^2-2F\eta'\zeta'-C\zeta'^2), & -\frac{2}{\sqrt{K}}(f\eta'-b\zeta')+\frac{2}{K}\eta'(A\xi'+H\eta'+G\zeta'), & -\frac{2}{\sqrt{K}}(c\eta'-f\zeta')+\frac{2}{K}\zeta'(A\xi'+H\eta'+G\zeta'), \\
-\frac{2}{\sqrt{K}}(a\zeta'-g\xi')+\frac{2}{K}\xi'(H\xi'+B\eta'+F\zeta'), & 1-\frac{2}{\sqrt{K}}(h\zeta'-f\eta')+\frac{1}{K}(B\eta'^2-C\zeta'^2-2G\zeta'\xi'-A\xi'^2), & -\frac{2}{\sqrt{K}}(g\zeta'-c\xi')+\frac{2}{K}\zeta'(H\xi'+B\eta'+F\zeta'), \\
-\frac{2}{\sqrt{K}}(h\xi'-a\eta')+\frac{2}{K}\xi'(G\xi'+F\eta'+C\zeta'), & -\frac{2}{\sqrt{K}}(b\xi'-h\eta')+\frac{2}{K}\eta'(G\xi'+F\eta'+C\zeta'), & 1-\frac{2}{\sqrt{K}}(f\xi'-g\eta')+\frac{1}{K}(C\zeta'^2-A\xi'^2-2H\xi'\eta'-B\eta'^2),
\end{array}
$$

and considering (X, Y, Z) and $(X_{,}, Y_{,}, Z_{,})$ as quantities connected by the foregoing linear relations, we have identically

$$(a, \ldots \between X, Y, Z)^2=(a, \ldots \between X_{,}, Y_{,}, Z_{,})^2.$$

So that the investigation leads to the automorphic transformation of the quadric function, a transformation first effected by M. HERMITE*.

33. It is to be remarked that the foregoing formulæ show that (x', y', z') being the coordinates of a point on the conic $(a, \ldots \between x, y, z)^2+(\xi'x+\eta'y+\zeta'z)^2=0$, from which point tangents are drawn to the conic $(a, \ldots \between x, y, z)^2=0$, then the coordinates (x', y', z') enter *linearly* into the equations of the tangents, the ineunts (or points of contact), and the polar. And it may be added that the equation of the conic enveloped by the polar (that is, the polar conic of $(a, \ldots \between x, y, z)^2+(\xi'x+\eta'y+\zeta'z)^2=0$) has for its equation

$$\{K+(A, \ldots \between \xi', \eta', \zeta')^2\}(a, \ldots \between x, y, z)^2-K(\xi'x+\eta'y+\zeta'z)^2=0.$$

and that the coordinates of the point of contact of the polar with this conic are

$$x'+\frac{1}{K}(A\xi'+H\eta'+G\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$y'+\frac{1}{K}(H\xi'+B\eta'+F\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$z'+\frac{1}{K}(G\xi'+F\eta'+C\zeta')(\xi'x'+\eta'y'+\zeta'z');$$

so that (x', y', z') also enter linearly into the expressions for the coordinates of the last-mentioned point.

Article Nos. 34 to 37, relating to two conics.

34. Considering now the two conics

$$U=(a, b, c, f, g, h \between x, y, z)^2=0,$$

$$U=(a', b', c', f', g', h' \between x, y, z)^2=0;$$

Suppose that the conic

$$\theta U+\theta' U'=(\theta a+\theta' a', \ldots \between x, y, z)^2=0$$

represents a pair of lines.

* See my "Memoir on the Automorphic Transformation of a Bipartite Quadric Function," Phil. Trans. vol. cxlviii. (1858) pp. 39–46.

The condition for this is

$$\text{Disct. } (\theta a+\theta' a', \ldots \between x, y, z)^2=0,$$

which is

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \between \theta, \theta')^3=0,$$

where

$$\begin{aligned}
\mathfrak{A} &= \text{K},\\
\mathfrak{B} &= \text{A}a'+\text{B}b'+\text{C}c'+2\text{F}f'+2\text{G}g'+2\text{H}h',\\
\mathfrak{C} &= \text{A}'a+\text{B}'b+\text{C}'c+2\text{F}'f+2\text{G}'g+2\text{H}'h,\\
\mathfrak{D} &= \text{K}'
\end{aligned}$$

(the significations of K', A', B', C', F', G', H' being of course analogous to those of K, A, B, C, F, G, H). The three roots $\theta : \theta'$ correspond, it is clear, to the three pairs of lines which can be drawn through the intersections of the two conics.

35. The equation

$$\text{Disct. } (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \between \theta, \theta')^3=0,$$

which is of the fourth order in $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, and of the sixth order as regards (a, b, c, f, g, h) and (a', b', c', f', g', h') respectively, is the condition in order that the two conics may touch each other. Assuming that it is satisfied, the cubic equation in $\theta : \theta'$ has a pair of equal roots; or say there is a twofold root and a onefold root; the twofold root gives the pair of lines drawn from the point of contact to the other two points of intersection, the onefold root gives the pair made up of the common tangent and the line joining the other two points of intersection.

36. In particular, suppose that the two conics are

$$\begin{aligned}
2(\varrho x+\sigma y+\tau z)(\varrho' x+\sigma' y+\tau' z)&=0,\\
2(\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z)&=0;
\end{aligned}$$

so that

$$\begin{aligned}
(a, b, c, f, g, h)&=(2\varrho\varrho', 2\sigma\sigma', 2\tau\tau', \sigma\tau'+\sigma'\tau, \tau\varrho'+\tau'\varrho, \varrho\sigma'+\varrho'\sigma),\\
(a', b', c', f', g', h')&=(2\lambda\lambda', 2\mu\mu', 2\nu\nu', \mu\nu'+\mu'\nu, \nu\lambda'+\nu'\lambda, \lambda\mu'+\lambda'\mu),\\
(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H})&=-(\sigma\tau'-\sigma'\tau, \tau\varrho'-\tau'\varrho, \varrho\sigma'-\varrho'\sigma)^2,\\
(\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}')&=-(\mu\nu'-\mu'\nu, \nu\lambda'-\nu'\lambda, \lambda\mu'-\lambda'\mu)^2;
\end{aligned}$$

and thence also

$$\mathfrak{A}=\text{K} \quad = \quad 0,$$

$$\mathfrak{B}=\text{A}a'+\&\text{c.}=-2 \begin{vmatrix} \lambda, & \mu, & \nu \\ \varrho, & \sigma, & \tau \\ \varrho', & \sigma', & \tau' \end{vmatrix} \begin{vmatrix} \lambda', & \mu', & \nu' \\ \varrho, & \sigma, & \tau \\ \varrho', & \sigma', & \tau' \end{vmatrix},$$

$$\mathfrak{C}=\text{A}'a+\&\text{c.}=-2 \begin{vmatrix} \varrho, & \sigma, & \tau \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} \begin{vmatrix} \varrho', & \sigma', & \tau' \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix},$$

$$\mathfrak{D}=\text{K}' \quad = \quad 0;$$

and the equation in (θ, θ') is

$$\mathfrak{B}\theta+\mathfrak{C}\theta'=0;$$

so that, writing $\theta=\mathfrak{C}$, $\theta'=-\mathfrak{B}$, the equation of the pair of lines is

$$\begin{vmatrix} \rho, & \sigma, & \tau \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} \begin{vmatrix} \rho', & \sigma', & \tau' \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} (\rho x+\sigma y+\tau z)(\rho' x+\sigma' y+\tau' z)- \begin{vmatrix} \lambda, & \mu, & \nu \\ \rho, & \sigma, & \tau \\ \rho', & \sigma', & \tau' \end{vmatrix} \begin{vmatrix} \lambda', & \mu', & \nu' \\ \rho, & \sigma, & \tau \\ \rho', & \sigma', & \tau' \end{vmatrix} (\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z)=0;$$

and it is easy to see that the left-hand side does in fact break up into factors, and that the equation is

$$\begin{vmatrix} x, & y, & z \\ \mu\tau'-\nu\sigma', & \nu\rho'-\lambda\tau', & \lambda\sigma'-\mu\rho' \\ \sigma\nu'-\tau\mu', & \tau\lambda'-\rho\nu', & \rho\mu'-\sigma\lambda' \end{vmatrix} \begin{vmatrix} x, & y, & z \\ \mu\tau-\nu\rho, & \nu\rho-\lambda\tau, & \lambda\sigma-\mu\rho \\ \sigma'\nu'-\tau'\mu', & \tau'\lambda'-\rho'\nu', & \rho'\mu'-\sigma'\lambda' \end{vmatrix}=0,$$

which of course might have been obtained at once by means of the four points which are the intersection of each component line of the first conic by each component line of the second conic.

37. Suppose that the first conic is

$$(a, b, c, f, g, h \between x, y, z)^2=0,$$

while the second conic is the pair of lines

$$2(\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z)=0;$$

then putting, as before,

$$(\theta a+\theta'.2\lambda\lambda', .. \between x, y, z)^2=0,$$

we have

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \between \theta, \theta')^3=0,$$

where

$$\begin{aligned} \mathfrak{A} &= \mathrm{K}, \\ \mathfrak{B} &= 2(\mathrm{A, B, C, F, G, H} \between \lambda, \mu, \nu \between \lambda', \mu', \nu'), \\ \mathfrak{C} &= -(a, b, c, f, g, h \between \mu\nu'-\mu'\nu, \nu\lambda'-\nu'\lambda, \lambda\mu'-\lambda'\mu)^2, \\ \mathfrak{D} &= 0; \end{aligned}$$

and the equation in (θ, θ') is

$$\mathrm{K}\theta^2+2(\mathrm{A}, .. \between \lambda, \mu, \nu \between \lambda', \mu', \nu')\theta\theta'-(a, .. \between \mu\nu'-\mu'\nu, ..)^2\theta'^2=0,$$

which may be written

$$\begin{aligned} \{\mathrm{K}\theta+(\mathrm{A}, .. \between \lambda, \mu, \nu \between \lambda', \mu', \nu')\theta'\}^2 &= \{[(\mathrm{A}, .. \between \lambda, \mu, \nu \between \lambda', \mu', \nu')]^2+\mathrm{K}(a, .. \between \mu\nu'-\mu'\nu, ..)^2\}\theta'^2 \\ &= (\mathrm{A}, .. \between \lambda, \mu, \nu)^2.(\mathrm{A}, .. \between \lambda', \mu', \nu')^2.\theta'^2, \end{aligned}$$

that is,

$$\mathrm{K}\theta=[\pm\sqrt{(\mathrm{A}, .. \between \lambda, \mu, \nu)^2}\sqrt{(\mathrm{A}, .. \between \lambda', \mu', \nu')^2}-(\mathrm{A}, .. \between \lambda, \mu, \nu \between \lambda', \mu', \nu')]\theta';$$

or we may assume

$$\theta = \pm\sqrt{(\mathrm{A},..\between\lambda,\mu,\nu)^2}\sqrt{(\mathrm{A},..\between\lambda',\mu',\nu')^2}-(\mathrm{A},..\between\lambda,\mu,\nu\between\lambda',\mu',\nu'),\quad \theta'=\mathrm{K},$$

so that the conic

$$\{\pm\sqrt{(\mathrm{A},..\between\lambda,\mu,\nu)^2}\sqrt{(\mathrm{A},..\between\lambda',\mu',\nu')^2}-(\mathrm{A},..\between\lambda,\mu,\nu\between\lambda',\mu',\nu')\}(a,\dots\between x,y,z)^2$$
$$+2\mathrm{K}(\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z)=0$$

breaks up into a pair of lines.

Putting for shortness

$$\pm\sqrt{(\mathrm{A},..\between\lambda,\mu,\nu)^2}\sqrt{(\mathrm{A},..\between\lambda',\mu',\nu')^2}-(\mathrm{A},..\between\lambda,\mu,\nu\between\lambda',\mu',\nu')=\Omega,$$

the coefficients on the left-hand side of the equation are

$$(\Omega a+2\mathrm{K}\lambda\lambda',..\ \Omega f+\mathrm{K}(\mu\nu'+\mu'\nu),..),$$

whence, after all reductions, the inverse function is

$$\{(\mathrm{A},..\between\lambda,\mu,\nu\between\xi,\eta,\zeta)\sqrt{(\mathrm{A},..\between\lambda',\mu',\nu')^2}\mp(\mathrm{A},..\between\lambda',\mu',\nu'\between\xi,\eta,\zeta)\sqrt{(\mathrm{A},..\between\lambda,\mu,\nu)^2}\}^2,$$

and the remainder of the process of decomposition is effected without difficulty.

ADDITION, 18 December, 1862.

The formulæ II. and II. (bis) each of them give the tangents of the conic $(a,\dots\between x,y,z)^2=0$ at the ineunts of intersection with the line $\xi' x+\eta' y+\zeta' z=0$. A very elegant formula for these ineunts themselves was communicated to me by Mr. SPOTTISWOODE, and I have since found that the same or an equivalent formula is made use of by M. ARONHOLD in his recent valuable memoir, "Ueber eine neue algebraische Behandlungsweise der Integrale irrationaler Differentiale, &c.", Crelle, t. lxii. pp. 95–145 (1862). The formula is as follows, viz. for the conic and line,

$$(a,b,c,f,g,h\between x,y,z)^2=0$$
$$\xi' x+\eta' y+\zeta' z=0,$$

then

$$x:y:z=$$

$$(l\xi'+m\eta'+n\zeta')\frac{1}{2\sqrt{\varphi}}\frac{d\varphi}{d\xi'}+\eta'(gl+fm+cn)-\zeta'(hl+bm+fn)+l\sqrt{\varphi},$$

$$:(l\xi'+m\eta'+n\zeta')\frac{1}{2\sqrt{\varphi}}\frac{d\varphi}{d\eta'}+\zeta'(al+hm+gn)-\xi'(gl+fm+cn)+m\sqrt{\varphi},$$

$$:(l\xi'+m\eta'+n\zeta')\frac{1}{2\sqrt{\varpi}}\frac{d\varphi}{d\zeta'}+\xi'(hl+bm+fn)-\eta'(al+hm+gn)+n\sqrt{\varphi};$$

where

$$\phi = \begin{vmatrix} & \xi', & \eta', & \zeta' \\ \xi', & a, & h, & g \\ \eta', & h, & b, & f \\ \zeta', & g, & f, & c \end{vmatrix} = -(\mathrm{A, B, C, F, G, H} \between \xi', \eta', \zeta')^2$$

So that $\frac{1}{2}\frac{d\phi}{d\xi'}$, $\frac{1}{2}\frac{d\phi}{d\eta'}$, $\frac{1}{2}\frac{d\phi}{d\zeta'}$ are respectively

$$= -(\mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta'),\ -(\mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta')-\mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta',$$

and where l, m, n are supernumerary arbitrary quantities.